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A new proof of existence of equilibria in infinite normal form games

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ABSTRACT

In this note, we prove the existence of Nash equilibria in infinite normal form games with compact sets of strategies and continuous payoffs by constructing Nash mappings.

1. Introduction

In the two seminal papers [1,2], Nash introduced a concept of equilibrium points as a natural solution concept for non-cooperative games and established the existence of equilibria in all finite games; especially, the proof in [2] based directly on the Brouwer theorem is a considerable improvement over the earlier version in [1] based on Kakutani fixed point theorem. In fact, in Nash's unpublished Ph.D. Dissertation [3], there are two interpretations of equilibrium concept for non-cooperative games, one rationalistic and one mass-action. However, the former is "quite strongly a rationalistic and idealizing interpretation", while the latter is more realistic and appropriate.

The mass-action view, as described by Weibull in [4], suggests that "Nash equilibria could be identified as stationary, or perhaps dynamically stable, population states in dynamic models of boundedly rational strategy adaptation in large strategically interactions populations". In detail, consider a finite n-player game G: Let A_i be the pure strategy set of players position $i \in I = \{1, \dots, n\}$, S_i be its mixed strategy simplex, the expected payoff to player position i when a profile $s \in S = \Pi_{i\in I} S_i$ is played be denoted $\pi_i(s)$, while $\pi_{i\alpha}(s)$ denotes the payoff to player i when he uses pure strategy $\alpha \in A_i$ against the profile $s \in S$. Now let the game be played over and over again by individuals who are randomly drawn from infinitely large populations, one population for each player position i in the game, while a population state for any time t is then formally identical with a mixed strategy profile $s(t) \in S$. Then, if a population state s is stationary (i.e, $s(t) = s$ for all t, or $\dot{s}_{i\alpha}(t) = 0$ for all i, α) under the following dynamics:

$$\dot{s}_{i\alpha}(t) = \pi^+_{i\alpha}(s) - s_{i\alpha} \sum_{\beta} \pi^+_{i\beta}(s)$$

where $\pi^+_{i\alpha}(s) = \max\{\pi_{i\alpha} - \pi_i(s), 0\}$, then s constitutes a Nash equilibrium, that is,

$$\pi_{i\alpha}(s) = \max_{\beta\in A_i} \pi_{i\beta}(s) \quad (\forall i \in I).$$

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It is not unique that a continuous-time analogue of the iteration mapping $T : S \to S$ defined by $T_i(s) = s'_i$ for

$$s'_{i\alpha} = \frac{s_{i\alpha} + \pi^+_{i\alpha}(s)}{1 + \sum_\beta \pi^+_{i\beta}(s)} \quad (\forall i \in I, \alpha \in A_i),$$

now described as Nash mapping, introduced in Nash's [2] influential existence proof for equilibrium points is nothing else than the population dynamics given above.

Accordingly, in this note, for infinite normal formal games with compact sets of strategies and continuous payoffs, we construct Nash mappings and also prove the existence of Nash equilibria by the Tychonov fixed point theorem [5], while Glicksberg [6] constructed best reply correspondences and established the existence of Nash equilibria by his generalized Kakutani fixed point theorem.

2. Nash mapping and the existence of equilibria

We consider an n-person infinite normal formal game f, in which each player i has a compact metric space X_i of pure strategies (with metric d_i) and a real-valued continuous payoff function f_i over $X = \Pi_{i=1}^n X_i$.

For each player i, denote by S_i the space of mixed strategies, or probability measures on X_i, endowed with ω^* topology, which is a compact convex set of a locally convex linear space (see [7]). And denote by δ_{x_i} the mixed strategy which assigns probability 1 to a pure strategy $x_i \in X_i$. Let $S = \Pi_{i=1}^n S_i$ be the product space of mixed strategy profiles. For each player i, the expect utility function on S is defined as follows: for any $\mu = (\mu_1, \ldots, \mu_n) \in S$,

$$u_i(\mu) = \int_X f_i(x_1, x_2, \ldots, x_n)\mathrm{d}\mu_1\mathrm{d}\mu_2 \ldots \mathrm{d}\mu_n.$$

Clearly, u_i is continuous on S by Proposition 2.1 in [8]. A mixed strategy profile $\mu \in S$ is called an equilibrium of the infinite game f if for each i,

$$u_i(\mu) = \max_{\nu_i \in S_i} u_i(\nu_i, \mu_{-i}),$$

where the symbol $-i$ denotes "all players but i" given a player i.

Given an infinite game f, for each i, for any $\mu \in S$ and for any Borel subset B_i of X_i, define

$$N_i(\mu)(B_i) = \frac{\mu_i(B_i) + \int_{B_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}\mathrm{d}\nu_i}{1 + \int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}\mathrm{d}\nu_i},$$

where ν_i is a mixed profile in S_i satisfying

$$\nu_i(\{x_i^k\}) = \frac{1}{2^k}, \quad k = 1, 2, \ldots$$

for some fixed dense countable subset $D_i = \{x_i^k : k = 1, 2, \ldots\}$ of compact metric space X_i. Then $N_i(\mu)$ is a mixed strategy in S_i. If $N_f = (N_1, N_2, \ldots, N_n)$ then $N_f : S \to S$ is a well defined mapping.

Lemma 2.1. *The mapping N_f is continuous on S and has at least one fixed point.*

Proof. Since S is metrizable [7], we need only to prove that for any sequence $\{\mu^k \in S : k = 1, 2, \ldots\}$ converging to $\mu \in S$ (under ω^*), $\{N_f(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_f(\mu)$ (under ω^*), or for each i, $\{N_i(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_i(\mu)$ (under ω^*).

Let g_i be any real-valued continuous function on X_i. Then, for any k,

$$\int_{X_i} g_i \mathrm{d}N_i(\mu^k) = \frac{\int_{X_i} g_i \mathrm{d}\mu_i^k + \int_{X_i} g_i \max\{0, u_i(\delta_{x_i}, \mu^k_{-i}) - u_i(\mu^k)\}\mathrm{d}\nu_i}{1 + \int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu^k_{-i}) - u_i(\mu^k)\}\mathrm{d}\nu_i}$$

(cf. [9]). So the sequence $\{\int_{X_i} g_i \mathrm{d}N_i(\mu^k) : k = 1, 2, \ldots\}$ converges to

$$\frac{\int_{X_i} g_i \mathrm{d}\mu_i + \int_{X_i} g_i \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}\mathrm{d}\nu_i}{1 + \int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}\mathrm{d}\nu_i} = \int_{X_i} g_i \mathrm{d}N_i(\mu).$$

That is, $\{N_i(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_i(\mu)$ (under ω^*). Therefore, N_i is continuous on S, and thus N_f is continuous on S.

Since S is also a compact convex set of a locally convex linear space, by the Tychonov fixed point theorem [2], there exists at least one fixed point of N_f. $\square$

Lemma 2.2. *A mixed strategy profile μ is an equilibrium of infinite game f if and only if it is a fixed point of N_f.*

Proof. Clearly, if $\mu \in S$ is an equilibrium of the game f then it is a fixed point of N_f. Then, we need to prove that if μ is a fixed point of N_f then it is an equilibrium of the game f.

Assume that μ is a fixed point of N_f. We claim that for each i,

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}\nu_i = 0.$$

Suppose that it were not. Then, there would exist some i such that

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}\nu_i > 0.$$

Since μ is a fixed point of N_f, then for any Borel subset B_i of X_i,

$$\mu_i(B_i) = \frac{\int_{B_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}\nu_i}{\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}\nu_i} = \frac{\sum\limits_{k \in B_i \cap D_i} 2^{-k} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}}{\sum\limits_{k}^{\infty} 2^{-k} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}}.$$

It is clear that $\mu_i(D_i) = 1$, which also implies that if $\mu(x_i^k) > 0$ (here $x_i^k \in D_i$) then $u_i(\delta_{x_i}, \mu_{-i}) > u_i(\mu)$. Hence $u_i(\mu_i, \mu_{-i}) = \sum_{k=1}^{\infty} \mu(x_i^k) u_i(\delta_{x_i}, \mu_{-i}) > u_i(\mu)$, the desired contradiction.[1]

We now prove that μ is an equilibrium.

Since for each i,

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}\nu_i = 0,$$

by the definition of ν_i, it is easy to show that for any $x_i^k \in D_i$,

$$u_i(\delta_{x_i^k}, \mu_{-i}) - u_i(\mu) \le 0.$$

Since D_i is dense in X_i, then for any $x_i \in X_i$, there is a sequence $\{x_i^{k_m} \in D_i : m = 1, 2, \ldots\}$ converging to x_i (under d_i). Clearly, the corresponding mixed strategy sequence $\{\delta_{x_i^{k_m}} : m = 1, 2, \ldots\}$ converges to δ_{x_i} (under ω^*). Since u_i is continuous,

$$u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu) \le 0.$$

Therefore, $u_i(\mu) = \max_{\nu_i \in S_i} u_i(\nu_i, \mu_{-i})$, that is, μ is an equilibrium. $\square$

Theorem 2.1. *There exists at least one equilibrium of infinite game f.*

Proof. It follows from Lemmas 2.1 and 2.2.

Since the mapping N_f, which has properties of Lemmas 2.1 and 2.2, is similar to the mapping T which is constructed by Nash [2] to establish the existence of equilibria in a finite game, we call the mapping N_f a *Nash mapping* of the infinite game f. $\square$

Remark 2.1. Our method to construct a Nash mapping for an infinite game is also valid to construct a Nash mapping for a finite game, which could be of benefit to computing Nash equilibria of a finite game.

Remark 2.2. Since the Tychonov fixed point theorem is simpler than the Glicksberg fixed point theorem, our proof of Nash equilibria in infinite games given here is an improvement over the earlier version of Glicksberg in [6].[2]

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[1] A referee points out a mistake in our earlier proof and suggests this version.

[2] The same referee suggests another simpler proof: given $\epsilon > 0$, obtain using the continuity of u_i, $\delta > 0$ corresponding to ϵ and obtain a finite open cover $\{B_\delta\{x_i^k\}\}_k$ of X_i. Consider the game G where players can only choose mixed strategies supported on $\{x_i^k\}_k$. This game has a Nash equilibrium μ_ϵ and we have that $u_i(\mu_\epsilon) > u_i(\delta_{x_i}) - \epsilon$ for all $x_i \in X_i$. The sequence μ_j obtained by setting $\epsilon = 1/j$ has a convergent subsequence and its limit point is a Nash equilibrium of the original game.

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